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Exact similarity solutions to some nonlinear diffusion equations

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Abstract. We give new closed-form similarity solutions to N-dimensional radially symmetric nonlinear diffusion equations with power-law and exponential diffusivities. These solutions are generalisations of known similarity solutions.

1. Introduction

This paper is concerned with N-dimensional radially symmetric nonlinear diffusion equations of the form

$$\frac{\partial c}{\partial t}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}D(c)\frac{\partial c}{\partial r}\right) \tag{1.1}$$

(when $N=1$ we shall sometimes write x in place of r). In particular, we shall consider the cases $D(c)=c^n$, so that

$$\frac{\partial c}{\partial t}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}c^n\frac{\partial c}{\partial r}\right) \tag{1.2}$$

and $D(c)=\mathrm{e}^c$, so that

$$\frac{\partial c}{\partial t}=\frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\,\mathrm{e}^c\frac{\partial c}{\partial r}\right). \tag{1.3}$$

Equations of the form (1.2), especially, have a large number of applications, for both $n>0$ ('slow' diffusion) and $n<0$ ('fast' diffusion); see, for example, [1] and [2]. Large numbers of exact similarity solutions to (1.2) are already known (most have been listed in [3], where some new forms were also determined; other recent results were given in [4]).

Here we shall determine new results by generalising known instantaneous source and dipole solutions to (1.2), after which we determine the corresponding solutions to (1.3). We then give some extensions to these results, and we conclude with some discussion.

2. Instantaneous source-type solutions

2.1. Introduction

The instantaneous source solution to (1.2) due to Barenblatt [5] and Pattle [6] takes the form, for $n \neq -2/N$, of

$$c = t^{-N/(nN+2)} f(r/t^{1/(nN+2)}) \tag{2.1}$$

while for $n = -2/N$ we may write

$$c = \mathrm{e}^{-\lambda N t} f(r/\mathrm{e}^{\lambda t}) \tag{2.2}$$

where λ is an arbitrary constant.

These forms of similarity variable are chosen so that the total mass Q defined by

$$Q = \int_0^\infty r^{N-1} c \, \mathrm{d}r$$

is fixed in time when the integral exists.

Substituting (2.1) or (2.2) into (1.2) gives the ordinary differential equation

$$-\lambda\left(N\eta^{N-1}f + \eta^N \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{N-1} f^n \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) \tag{2.3}$$

where $\lambda = 1/(nN+2)$ corresponds to (2.1). Integrating (2.3) gives

$$-\lambda\eta^N f = \eta^{N-1} f^n \frac{\mathrm{d}f}{\mathrm{d}\eta} + \alpha \tag{2.4}$$

where here and throughout the remainder α is an arbitrary constant. The usual instantaneous source solution corresponds to $\alpha = 0$ and then either $f = 0$ or

$$f = \left(\frac{\lambda n}{2}(\beta - \eta^2)\right)^{1/n} \qquad n \neq 0$$

$$f = \beta\, \mathrm{e}^{-\lambda\eta^2/2} \qquad n = 0$$

where here and henceforth β is a further arbitrary constant.

The corresponding solutions for c are best written in the form (assuming $t > 0$ in the cases (a)-(c)):

(a) $n > 0$

$$c = t^{-N/(nN+2)}\left[\frac{n}{2(nN+2)}\left(a^2 - \frac{r^2}{t^{2/(nN+2)}}\right)\right]^{1/n} \qquad r < at^{1/(nN+2)}$$

$$= 0 \qquad r \geq at^{1/(nN+2)}$$

where $\beta = a^2$. This solution for 'slow' diffusion has compact support and is the form given in [5] and [6].

(b) $n = 0$

$$c = At^{-N/2}\, \mathrm{e}^{-r^2/4t}$$

where $\beta = A$.

(c) $0 > n > -2/N$

$$c = t^{-N/(nN+2)}\left[\frac{-n}{2(nN+2)}\left(a^2 + \frac{r^2}{t^{2/(nN+2)}}\right)\right]^{1/n}$$

where $\beta = -a^2$.

(d) $n = -2/N$

$$c = \mathrm{e}^{-\lambda N t}\left[\frac{\lambda}{N}\left(a^2 + \frac{r^2}{\mathrm{e}^{2\lambda t}}\right)\right]^{-N/2}$$

where $\beta = -a^2$. In contrast to the previous cases the total mass Q associated with this solution is unbounded.

(e) $n < -2/N$

$$c = (-t)^{-N/(nN+2)}\left[\frac{-n}{-2(nN+2)}\left(a^2 + \frac{r^2}{(-t)^{2/(nN+2)}}\right)\right]^{1/n} \qquad t<0$$

$$= 0 \qquad t \geqslant 0.$$

The total mass associated with the solution for $t<0$ is unbounded, but c vanishes everywhere at $t=0$. This is a consequence of the way the diffusivity c^n blows up as $c \to 0$. We note that (1.1) is invariant under translations of t, so that the origin of time can be chosen arbitrarily.

The solutions for the cases (a)-(c) are well known and have found wide physical application.

We now return to (2.4) and obtain new solutions by considering the case $\alpha \neq 0$ for particular values of n and N.

2.2. *The case* $n = -1$

Equation (2.4) is then a Bernoulli equation that we solve by writing $f = 1/g$ to give

$$\frac{\mathrm{d}g}{\mathrm{d}\eta} - \alpha\eta^{1-N} g = \lambda\eta. \tag{2.5}$$

When $\alpha \neq 0$, $N \neq 2$, the solution to (2.5) is given by

$$g = \lambda \exp\left(\frac{\alpha\eta^{2-N}}{2-N}\right)\int_{\eta_0}^{\eta} \eta \exp\left(-\frac{\alpha\eta^{2-N}}{2-N}\right)\mathrm{d}\eta$$

where η_0 is an arbitrary constant.

For $N=1$ we have

$$g = \beta\,\mathrm{e}^{\alpha\eta} - \frac{\lambda}{\alpha^2}(1+\alpha\eta).$$

When $\beta = 0$ this gives a travelling wave solution

$$c = \alpha^2/[\lambda(\alpha x + t)].$$

For $N=3$

$$g = \beta\,\mathrm{e}^{-\alpha/\eta} + \frac{\lambda\eta}{2}(\alpha+\eta) + \frac{\lambda\alpha^2}{2}\mathrm{e}^{-\alpha/\eta}E_1\left(-\frac{\alpha}{\eta}\right)$$

where

$$E_1(z) = \int_z^{\infty}\frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t$$

is the exponential integral.

Finally, for $N=2$,

$$g = \beta\eta^{\alpha} + \frac{\lambda}{2-\alpha}\eta^2 \qquad \text{if } \alpha \neq 2$$

and

$$g = \beta\eta^2 + \lambda\eta^2 \ln\eta \qquad \text{if } \alpha = 2.$$

2.3. The case $n = -2/N$

Here (2.4) is invariant under the rescaling group $\eta \to \beta\eta$, $f \to \beta^{-N} f$; the case $n = -2$, $N = 1$ has been solved in [7]. The substitutions

$$f = \eta^{-N} g \qquad \xi = \ln \eta$$

transform (2.4) with $n = -2/N$ into

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = Ng - \alpha g^{2/N} - \lambda g^{1+2/N}$$

so that g is given by

$$\int_{g_0}^{g} \frac{\mathrm{d}g}{Ng - \alpha g^{2/N} - \lambda g^{1+2/N}} = \xi$$

where g_0 is an arbitrary constant.

Corresponding to the case

$$Ng_0 - \alpha g_0^{2/N} - \lambda g_0^{1+2/N} = 0$$

we have the solution

$$g = g_0$$

which gives the steady state solution

$$c = g_0 r^{-N}.$$

When $N = 2$ we have $n = -1$, which is included in the case given in section 2.2 above.

2.4. The case $n = -\frac{1}{2}$

If $n = -\frac{1}{2}$ then we write $f = g^2$ to obtain the Riccati equation

$$2\eta^{N-1} \frac{\mathrm{d}g}{\mathrm{d}\eta} + \lambda \eta^N g^2 = -\alpha \tag{2.6}$$

which we transform to a linear equation by the substitution

$$g = \frac{2}{\lambda \eta q} \frac{\mathrm{d}q}{\mathrm{d}\eta}$$

to give

$$\eta \frac{\mathrm{d}^2 q}{\mathrm{d}\eta^2} - \frac{\mathrm{d}q}{\mathrm{d}\eta} + \frac{\lambda\alpha}{4} \eta^{3-N} q = 0. \tag{2.7}$$

$N = 4$ corresponds to the case given in section 2.3 and equation (2.7) is an Euler equation; otherwise we write

$$q = \eta y \qquad z = \frac{(\lambda\alpha)^{1/2}}{4 - N} \eta^{(4-N)/2}$$

to obtain

$$z^2 \frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + z \frac{\mathrm{d}y}{\mathrm{d}z} + \left(z^2 - \frac{4}{(4-N)^2} \right) y = 0$$

which is a Bessel equation with general solution

$$y = AJ_\nu(z) + BY_\nu(z)$$

where A and B are arbitrary constants and $\nu = 2/(4-N)$. The general solution to (2.6) may then be written in the form:

$$g = \left(\frac{\alpha}{\lambda}\right)^{1/2} \eta^{-N/2} \left[\beta J_{\nu-1}\left(\frac{(\lambda\alpha)^{1/2}}{4-N}\,\eta^{(4-N)/2}\right) + (1-\beta) Y_{\nu-1}\left(\frac{(\lambda\alpha)^{1/2}}{4-N}\,\eta^{(4-N)/2}\right)\right]$$
$$\times \left[\beta J_\nu\left(\frac{(\lambda\alpha)^{1/2}}{4-N}\,\eta^{(4-N)/2}\right) + (1-\beta) Y_\nu\left(\frac{(\lambda\alpha)^{1/2}}{4-N}\,\eta^{(4-N)/2}\right)\right]^{-1}. \tag{2.8}$$

3. 'Dipole'-type solutions

3.1. Introduction

We now consider solutions that generalise the one-dimensional dipole solutions given for $n > 0$ in [8]. Solutions of this form in higher dimensions were referred to in [3] and [4] but few of the details were worked out. The similarity variables are chosen to fix

$$\int_0^\infty rc\, \mathrm{d}r$$

in time (if the integral exists). In one dimension this corresponds to the centre of mass.

The appropriate similarity solution to (1.2) then takes the form

$$c = t^{-1/(n+1)} f(r/t^{1/[2(n+1)]}) \qquad \text{if } n \neq 1 \tag{3.1}$$

with

$$c = \mathrm{e}^{-2\lambda t} f(r/\mathrm{e}^{\lambda t}) \qquad \text{if } n = -1 \tag{3.2}$$

where λ is an arbitrary constant. For $N = 2$ these solutions are the same as those of the previous section so we do not discuss $N = 2$ further.

For $n \neq -1$, f satisfies the ordinary differential equation

$$-\frac{1}{2(n+1)}\left(2\eta f + \eta^2 \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = \eta \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^n \frac{\mathrm{d}f}{\mathrm{d}\eta}\right) + (N-1) f^n \frac{\mathrm{d}f}{\mathrm{d}\eta}$$

giving

$$-\frac{1}{2(n+1)}\eta^2 f = \eta f^n \frac{\mathrm{d}f}{\mathrm{d}\eta} + (N-2)\frac{f^{n+1}}{n+1} + \alpha. \tag{3.3}$$

When $\alpha = 0$ (3.3) is a Bernoulli equation if $n \neq 0$ and may be solved in closed form to give

$$f = \eta^{(2-N)/(n+1)}\left[\frac{n}{2(nN+2)}\left(\beta - \eta^{(nN+2)/(n+1)}\right)\right]^{1/n} \qquad n \neq 0 \text{ and } n \neq -\frac{2}{N}$$
$$f = \beta \eta^{2-N}\, \mathrm{e}^{-\eta^2/4} \qquad n = 0 \tag{3.4}$$
$$f = \left(\frac{1}{N-2}\eta^2 \ln\frac{\eta}{\beta}\right)^{-N/2} \qquad n = -\frac{2}{N}.$$

The best form in which to write the solution for c again depends on the values of n and N. Since N is positive we may write (omitting the special cases $n=0$ and $n=-2/N$):

(a) $n>0$

$$c = t^{-1/(n+1)}\left(\frac{r}{t^{1/[2(n+1)]}}\right)^{(2-N)/(n+1)} \times \left\{\frac{n}{2(nN+2)}\left[a^{(nN+2)/(n+1)} - \left(\frac{r}{t^{1/[2(n+1)]}}\right)^{(nN+2)/(n+1)}\right]\right\}^{1/n} \qquad r < a t^{1/[2(n+1)]}$$
$$= 0 \qquad r \geqslant a t^{1/[2(n+1)]}. \tag{3.5}$$

This solution again has compact support, but

$$c \to \infty \qquad \text{as } r \to 0 \text{ if } N > 2.$$

(b) $0 > n > -2/N$

$$c = t^{-1/(n+1)}\left(\frac{r}{t^{1/[2(n+1)]}}\right)^{(2-N)/(n+1)} \times \left\{\frac{-n}{2(nN+2)}\left[a^{(nN+2)/(n+1)} + \left(\frac{r}{t^{1/[2(n+1)]}}\right)^{(nN+2)/(n+1)}\right]\right\}^{1/n}. \tag{3.6}$$

At a fixed value of $\eta = r/t^{1/[2(n+1)]}$ this blows up as $t \to \infty$ if $n < -1$ (which requires $N < 2$) and the corresponding value of r goes to zero while for $n > -1$ c decays to zero for fixed η as $t \to \infty$.

Solutions (3.5) and (3.6) each have

$$c = \mathrm{O} r^{(2-N)/(n+1)} \qquad \text{as } r \to 0$$

so that in (3.6), $c \to \infty$ as $r \to 0$ if $N > 2$ or if $n < -1$. Expression (3.6) also has

$$c = \mathrm{O}(r^{2/n}) \qquad \text{as } r \to \infty.$$

(c) $n < -2/N$

$$c = (-t)^{-1/(n+1)}\left(\frac{r}{(-t)^{1/[2(n+1)]}}\right)^{(2-N)/(n+1)} \times \left\{\frac{-n}{-2(nN+2)}\left[a^{(nN+2)/(n+1)} + \left(\frac{r}{(-t)^{1/[2(n+1)]}}\right)^{(nN+2)/(n+1)}\right]\right\}^{1/n} \qquad t < 0$$
$$= 0 \qquad t \geqslant 0. \tag{3.7}$$

This solution extinguishes at finite time ($t=0$) if $n < -1$ and blows up at $t=0$ if $0 > n > -1$ (which requires $N > 2$); in the latter case r goes to zero for any fixed value of

$$\eta = r(-t)^{-1/[2(n+1)]}.$$

In (3.7), c blows up as $r \to 0$ if $N < 2$ or if $n > -1$.

We note from (3.4)-(3.6) that since

$$c = \mathrm{O}(r^{(2-N)/(n+1)}) \qquad \text{as } r \to 0$$

and, if $n<0$,

$$c=\mathrm{O}(r^{2/n}) \qquad \text{as } r\to\infty$$

the conditions necessary for $\int_0^\infty rc\,\mathrm{d}r$ to be bounded are that $n>-1$ and $n>\frac{1}{2}N-2$. For the total mass $\int_0^\infty r^{N-1}c\,\mathrm{d}r$ to be bounded we need $n>-2/N$ and $n>-1$.

Finally, we note that for $n=-2/N$ the solution (3.4) blows up both at $\eta=0$ and at $\eta=\beta$.

The other special case we need to discuss is $n=-1$ so that (3.2) holds and

$$-\lambda\eta^2 f=\eta f^{-1}\frac{\mathrm{d}f}{\mathrm{d}\eta}+(N-2)\ln f+\alpha$$

where α is an arbitrary constant. By the rescalings

$$f\to \mathrm{e}^{-\alpha/(N-2)}f \qquad \eta\to\lambda^{-1/2}\,\mathrm{e}^{\alpha/[2(N-2)]}\eta$$

we may without loss of generality consider

$$-\eta^2 f=\eta f^{-1}\frac{\mathrm{d}f}{\mathrm{d}\eta}+(N-2)\ln f. \tag{3.8}$$

Further progress towards an exact solution does not seem to be possible, although (3.8) may be written in a more compact form by introducing

$$f=\exp\left[-\left(\frac{\eta}{N^{1/2}}\right)^{2-N}q\right]$$

$$\eta=N^{1/2}\xi^{1/N}$$

to give

$$\frac{\mathrm{d}q}{\mathrm{d}\xi}=\exp(-\xi^{(2-N)/N}q). \tag{3.9}$$

For $N=1$, (3.9) is invariant under two discrete transformations

$$\xi\to-\xi \qquad q\to-q$$

and

$$\xi\to\mathrm{i}q \qquad q\to\mathrm{i}\xi.$$

The first of these corresponds to the invariance of (3.8) under

$$\eta\to-\eta \qquad f\to f$$

which holds for any value of N.

We now return to (3.3) with $n\neq-1$ and with $\alpha\neq0$, and determine the general solution for particular values of n and N.

3.2. The case $n=-\frac{1}{2}$

We put $f=g^2$ so that for $n=-\frac{1}{2}$ (3.3) becomes the Riccati equation

$$2\eta\frac{\mathrm{d}g}{\mathrm{d}\eta}+2(N-2)g+\eta^2g^2=-\alpha$$

and writing $g=(2/\eta q)\,\mathrm{d}q/\mathrm{d}\eta$ gives

$$\eta\frac{\mathrm{d}^2q}{\mathrm{d}\eta^2}+(N-3)\frac{\mathrm{d}q}{\mathrm{d}\eta}+\frac{\alpha}{4}\eta q=0$$

so that

$$q=\eta^{(4-N)/2}\left[AJ_\nu\left(\frac{\alpha^{1/2}\eta}{2}\right)+BY_\nu\left(\frac{\alpha^{1/2}\eta}{2}\right)\right]$$

where A and B are arbitrary constants and $\nu=2-\frac{1}{2}N$. Hence

$$g=\frac{\alpha^{1/2}}{\eta}\left[\beta J_{\nu-1}\left(\frac{\alpha^{1/2}\eta}{2}\right)+(1-\beta)Y_{\nu-1}\left(\frac{\alpha^{1/2}\eta}{2}\right)\right]\left[\beta J_\nu\left(\frac{\alpha^{1/2}\eta}{2}\right)+(1-\beta)Y_\nu\left(\frac{\alpha^{1/2}\eta}{2}\right)\right]^{-1}. \tag{3.10}$$

For $N=2$, (3.10) is the same as (2.8), while for $N=3$ and $N=1$ (3.10) can be written in a much simpler form.

For $N=3$

$$g=-\frac{\alpha^{1/2}}{\eta}\tan\left(\frac{\alpha^{1/2}}{2}(\eta-\eta_0)\right)$$

or

$$g=\frac{\gamma^{1/2}}{\eta}\tanh\left(\frac{\gamma^{1/2}}{2}(\eta-\eta_0)\right)\qquad\text{(cf [4])}$$

where $\gamma=-\alpha$ and η_0 is an arbitrary constant.

For $N=1$, (3.10) becomes

$$g=\frac{\alpha}{2-\alpha^{1/2}\eta\cot[\frac{1}{2}\alpha^{1/2}(\eta-\eta_0)]} \tag{3.11}$$

or

$$g=-\frac{\gamma}{2-\gamma^{1/2}\eta\coth[\frac{1}{2}\gamma^{1/2}(\eta-\eta_0)]}\qquad \gamma=-\alpha. \tag{3.12}$$

Taking $\alpha\to 0$ with $\eta_0=-a^3\alpha/12$, (3.11) becomes

$$g=\frac{6\eta}{\eta^3+a^3}$$

the usual dipole solution, while if we take $\eta_0\to-\infty$ with $\gamma>0$, (3.12) becomes

$$g=\frac{\gamma}{\gamma^{1/2}\eta-2}$$

so that

$$c=\frac{4}{\mu^2(x-\mu t)^2}$$

where $\mu=2/\gamma^{1/2}$, which is a travelling wave solution.

3.3. *The case* $n=\frac{1}{2}N-2$

Writing

$$f=\eta^{(2-N)/(n+1)}g$$

(3.3) becomes

$$-\frac{1}{2(n+1)}\eta^{(2n+4-N)/(n+1)}g=\eta^{3-N}g^{n}\frac{\mathrm{d}g}{\mathrm{d}\eta}+\alpha. \tag{3.13}$$

When $n=\frac{1}{2}N-2$, (3.13) is separable and has solution

$$\int_{g_0}^{g}\frac{g^{\frac{1}{2}N-2}}{g+(N-2)\alpha}\,\mathrm{d}g=-\frac{1}{(N-2)^2}\eta^{N-2}$$

where g_0 is an arbitrary constant.

Corresponding to the case

$$g_0+(N-2)\alpha=0$$

we have $g=g_0$ giving the steady state solution

$$c=g_0r^2.$$

For $N=3$ the solution of this section corresponds to that of section 3.2.

4. $D(c)=\exp(c)$

We now turn to equation (1.3). If we write $u=\mathrm{e}^c$ then (1.3) becomes

$$\frac{\partial u}{\partial t}=\frac{u}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial u}{\partial r}\right)$$

whereas if we write $u=c^n$, (1.2) becomes

$$\frac{\partial u}{\partial t}=\frac{u}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial u}{\partial r}\right)+\frac{1}{n}\left(\frac{\partial u}{\partial r}\right)^2.$$

This indicates that (1.3) is related to (1.2) in the limit $n\to\infty$. It is therefore of interest to determine the solutions of (1.3) that correspond to the limit as $n\to\infty$ of the solutions we have already calculated. For both instantaneous source and dipole solutions the relevant limit gives a similarity solution to (1.3) of the form

$$c=-\ln t+f(r)$$

so that

$$-1=\frac{1}{r^{N-1}}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{N-1}\mathrm{e}^{f}\frac{\mathrm{d}f}{\mathrm{d}r}\right)$$

and

$$\mathrm{e}^{f}=-\frac{1}{2N}r^2+\alpha+\beta r^{2-N}\qquad\text{if } N\neq 2$$

and

$$\mathrm{e}^{f}=-\tfrac{1}{4}r^2+\alpha+\beta\ln r\qquad\text{if } N=2.$$

The solution corresponding to $N=1$ has already been given in [3, 9].

5. Some extensions

It is well known [10, 11] that for $N=1$ (1.2) may be mapped by a generalised Bäcklund transformation into the closely related equation

$$\frac{\partial C}{\partial T}=\frac{\partial}{\partial X}\left(C^{-2-n}\frac{\partial C}{\partial X}\right). \tag{5.1}$$

We make the transformation in two steps. Writing

$$c=\partial v/\partial x$$

the equation

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left(c^{n}\frac{\partial c}{\partial x}\right) \tag{5.2}$$

becomes

$$\frac{\partial v}{\partial t}=\left(\frac{\partial v}{\partial x}\right)^{n}\frac{\partial^2 v}{\partial x^2}. \tag{5.3}$$

We may note in passing that solutions to (5.3) equivalent to those of section 2 (with $\alpha=0$, [12]) and sections 2 and 3 (with $\alpha=0$ and $n=2$, [13]) have been directly determined apparently without recognising the equivalence to solutions of (5.2). Equation (5.3) has a wide class of similarity solutions equivalent to those of (5.2) including solutions of the form

$$v=t^{1/2}f(x/t^{1/2})$$

which is appropriate to constant inlet and outface flow rates for the problem discussed in [12], which, as mentioned in their discussion, could not be solved by the similarity variables introduced by the authors.

Introducing the hodograph-type transformation

$$V=x \qquad X=v \qquad T=t \tag{5.4}$$

equation (5.3) becomes

$$\frac{\partial V}{\partial T}=\left(\frac{\partial V}{\partial X}\right)^{-2-n}\frac{\partial^2 V}{\partial X^2} \tag{5.5}$$

and writing $C=\partial V/\partial X=1/c$ gives (5.1).

We now apply this transformation to the solutions of sections 2 and 3 in turn.

If $N=1$, (2.4) is

$$-\lambda\eta f=f^{n}\frac{\mathrm{d}f}{\mathrm{d}\eta}+\alpha$$

and writing $f=\mathrm{d}h/\mathrm{d}\eta$ gives

$$-\lambda\eta\frac{\mathrm{d}h}{\mathrm{d}\eta}=\left(\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)^{n}\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2}+\alpha$$

and this is equivalent to seeking a similarity solution to (5.3) of the form

$$v=-\alpha\ln t+h(x/t^{\lambda}) \qquad \lambda=1/(N+2) \qquad \text{if } n\neq-2$$

and

$$v = -\alpha t + h(x/e^{\lambda t}) \qquad \text{if } n = -2.$$

Applying the transformation (5.4) then gives

$$V = T^{\lambda} H(X + \alpha \ln T) \qquad n \neq -2$$

$$V = e^{\lambda T} H(X + \alpha T) \qquad n = -2$$

where $H(h(\eta)) = \eta$, and this gives

$$C = T^{\lambda} F(X + \alpha \ln T) \qquad n \neq -2$$

$$C = e^{\lambda T} F(X + \alpha T) \qquad n = -2$$

as a solution to (5.1) where $F(\eta) = \mathrm{d}H(\eta)/\mathrm{d}\eta$. Hence if $\alpha = 0$ we obtain the separable solution of (5.1) which can also be obtained directly (see, for example, [3]). For $\alpha \neq 0$ we have a similarity solution that has an application to dopant diffusion in silicon ([14]). We may now extend the results of section 2 to obtain some solutions for $\alpha \neq 0$.

(i) $n = -1$. Equations (5.1) and (5.2) then take the same form, and the solution

$$C = TF(X + \alpha \ln T)$$

can be obtained directly from (5.1). We have

$$F + \alpha \frac{\mathrm{d}F}{\mathrm{d}\eta} = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(F^{-1} \frac{\mathrm{d}F}{\mathrm{d}\eta}\right)$$

or, writing $F = \exp(p)$,

$$e^{p}\left(1 + \alpha \frac{\mathrm{d}p}{\mathrm{d}\eta}\right) = \frac{\mathrm{d}^2 p}{\mathrm{d}\eta^2}$$

which integrates to give

$$\alpha^2 e^{p} = \alpha \frac{\mathrm{d}p}{\mathrm{d}\eta} - \ln\left[\left(1 + \alpha \frac{\mathrm{d}p}{\mathrm{d}\eta}\right)\beta^{-1}\right]$$

which together with

$$\int_{q_0}^{\mathrm{d}p/\mathrm{d}\eta} \frac{\mathrm{d}q}{\{\alpha q - \ln[(1 + \alpha q)/\beta]\}(1 + \alpha q)} = \frac{\eta}{\alpha^2}$$

where q_0 is an arbitrary constant, gives the solution in implicit form. Corresponding to $\beta = 0$ we have the closed-form solution

$$p = p_0 - \eta/\alpha$$

giving the steady state solution

$$C = \exp(p_0 - X/\alpha).$$

(ii) $n = -2$. Equation (5.1) is then the linear diffusion equation and writing

$$C = e^{\lambda T} F(X + \alpha T)$$

gives

$$\frac{\mathrm{d}^2F}{\mathrm{d}\eta^2}=\alpha\frac{\mathrm{d}F}{\mathrm{d}\eta}+\lambda F$$

so that C is in general the sum of two travelling wave solutions.

(iii) $n=-\frac{1}{2}$. Equation (5.1) is now

$$\frac{\partial C}{\partial T}=\frac{\partial}{\partial X}\left(C^{-3/2}\frac{\partial C}{\partial X}\right)$$

and we are able to construct the general similarity solution of the form

$$C=T^{2/3}F(X+\alpha\ln T)$$

by using (2.8), which for $N=1$ may be written as

$$g=\left(\frac{3\alpha^2}{4}\right)^{1/3}\left(\frac{\gamma\mathrm{Ai}(-(\frac{1}{6}\alpha)^{1/3}\eta)+(1-\gamma)\mathrm{Bi}(-(\frac{1}{6}\alpha)^{1/3}\eta)}{\gamma\mathrm{Ai}'(-(\frac{1}{6}\alpha)^{1/3}\eta)+(1-\gamma)\mathrm{Bi}'(-(\frac{1}{6}\alpha)^{1/3}\eta)}\right) \tag{5.6}$$

where γ is an arbitrary constant and Ai and Bi are Airy functions with derivatives Ai′ and Bi′.

We construct F by writing

$$h(\eta)=\int_{\eta_0}^{\eta}g^2(\eta)\,\mathrm{d}\eta$$

where $g(\eta)=f^{1/2}(\eta)$ is given by (5.6) and η_0 is arbitrary. V is then given by

$$h\left(\frac{V}{T^{2/3}}\right)=X+\alpha\ln T \tag{5.7}$$

and, differentiating (5.7),

$$C=T^{2/3}/g^2(V/T^{2/3})$$

which together with (5.7) is an implicit solution for C. We have thus determined the general solution of the equation

$$\tfrac{2}{3}F+\alpha\frac{\mathrm{d}F}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(F^{-3/2}\frac{\mathrm{d}F}{\mathrm{d}\eta}\right).$$

We now turn to extensions of the dipole solutions of section 3. For $N=1$ we have if $n\neq-1$

$$-\frac{1}{2(n+1)}\left(2f+\eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^n\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)$$

and writing $f=\mathrm{d}h/\mathrm{d}\eta$ gives

$$-\frac{1}{2(n+1)}\left(h+\eta\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)=\left(\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)^n\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2} \tag{5.8}$$

(the constant of integration may be taken to be zero without loss of generality) so that

$$-\frac{1}{2(n+1)}\eta h=\frac{1}{n+1}\left(\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)^{n+1}-\alpha. \tag{5.9}$$

The constant of integration must be chosen as $-\alpha$ to make (5.8) and (5.9) consistent with (3.3).

If $\alpha = 0$ we have

$$h = -2\left[\frac{n}{2(n+2)}(\beta - \eta^{(n+2)/(n+1)})\right]^{(n+1)/n} \qquad n \neq 0 \qquad n \neq -2$$

$$h = -2\beta\, \mathrm{e}^{-\eta^2/4} \qquad n = 0$$

$$h = [-4\ln(\eta/\beta)]^{1/2} \qquad n = -2$$

which is equivalent to (3.4).

The corresponding similarity solution of (5.3) takes the form

$$v = t^{-1/[2(n+1)]}h(x/t^{1/[2(n+1)]})$$

and maps into the equivalent solution of (5.5), that is

$$V = T^{1/[2(n+1)]}H(XT^{1/[2(n+1)]})$$

where $H(h(\eta)) = \eta$.

We may therefore use the solution (3.11) to obtain the general dipole solution to

$$\frac{\partial C}{\partial T} = \frac{\partial}{\partial X}\left(C^{-3/2}\frac{\partial C}{\partial X}\right)$$

as follows (this solution also corresponds to that of section 3.3 with $N = 1$).

We calculate h from

$$h(\eta) = \int_{\eta_0}^{\eta} g^2(\eta)\, \mathrm{d}\eta$$

with $g(\eta)$ given by (3.11). V is then given by

$$h(V/T) = XT \tag{5.10}$$

so C is determined by (5.10) and

$$C = T^2/g^2(V/T).$$

We remark that (5.9) is a Riccati equation for h when $n = -\frac{1}{2}$; when $n = -\frac{3}{2}$ it is a Riccati equation for $\eta(h)$, which provides an alternative means for the derivation of this solution. More generally, the transformation $h \to \eta$, $\eta \to h$ (which is essentially (5.4)) maps (5.9) into another equation of the same type. When $n = -2$, (5.4) is then mapped into a linear equation so that the general solution may be found.

Finally, if $n = -1$ we have

$$-\lambda\left(2f + \eta\frac{\mathrm{d}f}{\mathrm{d}\eta}\right) = \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f^{-1}\frac{\mathrm{d}f}{\mathrm{d}\eta}\right)$$

so writing $f = \mathrm{d}h/\mathrm{d}\eta$ gives

$$-\lambda\left(h + \eta\frac{\mathrm{d}h}{\mathrm{d}\eta}\right) = \left(\frac{\mathrm{d}h}{\mathrm{d}\eta}\right)^{-1}\frac{\mathrm{d}^2h}{\mathrm{d}\eta^2}$$

so that

$$\frac{\mathrm{d}h}{\mathrm{d}\eta} = \alpha\, \mathrm{e}^{-\lambda\eta h}$$

where α is chosen to be consistent with section 4. This is equivalent to (3.9) with $N=1$ (to obtain (3.9) for $N=1$ we write $f=\exp(\eta q)$ which implies that $f=\mathrm{d}q/\mathrm{d}\eta$) and the second discrete invariant of (3.9) arises from the well known (for example, [15]) discrete invariant of

$$\frac{\partial v}{\partial t}=\left(\frac{\partial v}{\partial x}\right)^{-1}\frac{\partial^2 v}{\partial x^2}$$

given by $v\to x$ and $x\to v$.

A number of further generalisations in one dimension are possible; by the transformations described in [9] solutions of (1.2) may be used to give solutions of

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left((c+c_1)^n(c+c_2)^{-2-n}\frac{\partial c}{\partial x}\right)$$

where c_1 and c_2 are constants, and solutions of (1.3) can be transformed to solutions of

$$\frac{\partial c}{\partial t}=\frac{\partial}{\partial x}\left((c+c_1)^{-2}\,\mathrm{e}^{-\alpha/(c+c_1)}\frac{\partial c}{\partial x}\right).$$

6. Discussion

We have derived a number of exact solutions of nonlinear diffusion equations, and some of these can be used to illustrate a number of unusual effects. In particular, the solutions (3.5)–(3.7) have the following behaviour (with $n\neq -2/N$, $n\neq -1$, $N\neq 2$):

(I) As $r\to 0$, $c=O(r^{(2-N)/(n+1)})$. The local behaviour near $r=0$ is therefore quasi-steady. If $N>2$, $n>-1$ or $N<2$, $n<-1$ we have $c\to\infty$ as $r\to 0^+$; otherwise $c\to 0$ as $r\to 0^+$.

(II) For $n>-2/N$ the behaviour as $t\to+\infty$ is as follows:

at fixed η,

$$c\to 0 \qquad \text{as } t\to\infty \qquad \text{if } n>-1$$
$$c\to\infty \qquad \text{as } t\to\infty \qquad \text{if } n<-1.$$

at fixed $r>0$,

$$\eta\to 0 \qquad \text{as } t\to\infty \qquad \text{if } n>-1$$
$$\eta\to\infty \qquad \text{as } t\to\infty \qquad \text{if } n<-1$$

and

$$c\sim r^{(2-N)/(n+1)}t^{-(2n+4-N)/2(n+1)^2} \qquad \text{as } t\to\infty \qquad \text{if } n>-1$$
$$c\sim r^{2/n}t^{-1/n} \qquad \text{as } t\to\infty \qquad \text{if } n<-1$$

so that at fixed $r>0$, c blows up as $t\to\infty$ if $n<-1$ or if $n>-1$ and $N>2(n+2)$, and c decays to zero as $t\to\infty$ if $n>-1$ and $N<2(n+2)$.

(III) For $n<-2/N$ the behaviour as $t\to 0^-$ is as follows:

at fixed η,

$$c\to\infty \qquad \text{as } t\to 0^- \qquad \text{if } n>-1$$
$$c\to 0 \qquad \text{as } t\to 0^- \qquad \text{if } n<-1$$

at fixed $r>0$,

$$\eta\to\infty \qquad \text{as } t\to 0^- \qquad \text{if } n>-1$$

$$\eta\to 0 \qquad \text{as } t\to 0^- \qquad \text{if } n<-1$$

and

$$c\sim r^{2/n}(-t)^{-1/n} \qquad \text{as } t\to 0^- \qquad \text{if } n>-1$$

$$c\sim r^{(2-N)/(n+1)}(-t)^{-(2n+4-N)/2(n+1)^2} \qquad \text{as } t\to 0^- \qquad \text{if } n<-1$$

so that at fixed $r>0$, c blows up as $t\to 0^-$ if $n<-1$ and $N<2(n+2)$, and c vanishes as $t\to 0^-$ if $n>-1$ or if $n<-1$ and $N>2(n+2)$.

We note that the two cases in which c blows up at fixed η are cases in which $c\to\infty$ as $r\to 0^+$ for all time. They are also the two cases in which the behaviour in time is qualitatively different from the solutions of section 2 with $\alpha=0$; in the latter case $c\to 0$ as $t\to+\infty$ if $n>-2/N$ and $c\to 0$ as $t\to 0^-$ if $n<-2/N$. In section 2 $n=-2/N$ is the important dividing line; in the dipole case $n=-1$, $N=2$ and $N=2(n+2)$ are also of importance.

These four dividing lines cross at $n=-1$, $N=2$ and there is a simple change of variable that maps this case into the one-dimensional case. Given

$$\frac{\partial c}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(rc^{-1}\frac{\partial c}{\partial r}\right)$$

we write

$$u=r^2c \qquad x=\ln r$$

to obtain

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u^{-1}\frac{\partial u}{\partial x}\right).$$

Hence for $n=-1$ the one-dimensional solutions can be mapped into radially symmetric solutions and vice versa. In particular, in this paper we have constructed the general similarity solutions of the form (in the current notation)

$$u=t^{-1}f(x/t) \qquad u=tf(x+\alpha\ln t)$$

and these map, respectively, to the cylindrically symmetric solutions

$$c=r^{-2}t^{-1}f((\ln r)/t) \qquad c=t^{1-2\alpha}\hat{f}(r/t^{\alpha})$$

where $\hat{f}(\eta)=\eta^{-2}f(\ln\eta)$; the latter case encompasses a whole class of similarity solutions. We have also given the first integral of the ordinary differential equation corresponding to

$$u=e^{-2\lambda t}f(x/e^{\lambda t})$$

and this maps to

$$c=r^{-2}\,e^{-2\lambda t}f(\ln r/e^{\lambda t}).$$

Finally, we have constructed the general cylindrical similarity solution

$$c=e^{-\lambda t}f(r/e^{\lambda t})$$

and this maps to the travelling wave solution

$$u = \hat{f}(x - \lambda t)$$

where

$$\hat{f}(\eta) = e^{2\eta} f(e^{\eta}).$$

We summarise our results by listing the cases for which we have obtained the most general similarity solution of the given form (λ and α are arbitrary constants).

(i) Solutions to equation (1.2) in the following special cases:

(*a*)

$n = -\frac{1}{2}$	$c = t^{-(2N)/(4-N)} f(r/t^{2/(4-N)})$		section 2.4
	$c = t^{-2} f(r/t)$		section 3.2

(*b*)

$n = -1$	$c = t^{-N/(2-N)} f(r/t^{1/(2-N)})$	$N \neq 2$	section 2.2
	$c = tf(r + \alpha \ln t)$	$N = 1$	section 5, (i)
	$c = e^{-2\lambda t} f(r/e^{\lambda t})$	$N = 2$	section 2.2
	$c = r^{-2} t^{-1} f((\ln r)/t)$	$N = 2$	section 6
	$c = t^{1-2\alpha} f(r/t^{\alpha})$	$N = 2$	section 6

(*c*)

$n = -\frac{3}{2}$	$c = t^{2/3} f(r + \alpha \ln t)$	$N = 1$	section 5, (iii)
	$c = t^2 f(rt)$	$N = 1$	section 5, (5.10)

(*d*)

$n = -2/N$	$c = e^{-\lambda N t} f(r/e^{\lambda t})$		section 2.3

(*e*)

$n = \frac{1}{2}N - 2$	$c = t^{-2/(N-2)} f(r/t^{1/(N-2)})$	$N \neq 2$	section 3.3.

(ii) Solutions to equation (1.3) of the form

$$c = -\ln t + f(r) \qquad \text{section 4.}$$

The most general similarity solutions that we have been able to construct for (1.2) are all for 'fast' diffusion cases with $n = -1$, $n = -2/N$, $n = \frac{1}{2}N - 2$, $n = -\frac{1}{2}$ and $n = -\frac{3}{2}$. While this paper is not concerned with the applications of these solutions it is worth pointing out that diffusivities like these arise in a large number of contexts. For example, $n = -\frac{1}{2}$ arises in plasma diffusion [16] and in the thermal expulsion of liquid helium [17].

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